Supplemental Digital Appendix 1

Specific Equations Used to Formulate the Dynamic Measurement Model

Mixed Effects Modeling and DMM

The mixed –effects model utilized to formulate the DMM in this study can be expressed as Equation 1 below[1]:

$$y_{ti} = f(\boldsymbol{\phi}_i, \boldsymbol{x}_{tj}) + e_{ij} \tag{1}$$

Where y_{ti} is the response at time *t* for person *i*, $\boldsymbol{x}_{tj}$ is the predictor vector for the *t*th response for person *i*, $\boldsymbol{\phi}_i$ is the parameter vector for the *i*th person, *f* is a function of the predictor vector and the parameter vector, and e_{ij} is the within-person residual at time *t* for person *i*. The parameter vector varies from person to person, which can be incorporated by expanding $\boldsymbol{\phi}_i$ as

$$\boldsymbol{\phi}_i = \boldsymbol{X}_i\boldsymbol{\beta} + \boldsymbol{Z}_i\boldsymbol{b}_i,\ \boldsymbol{b}_i \sim N(\mathbf{0}, \mathbf{G}) \tag{2}$$

Where $\boldsymbol{\beta}$ is a vector of fixed effect coefficients that capture the marginal association of predictors, $\boldsymbol{b}_i$ is a person-specific vector of random effects capturing an individual's deviation from the marginal effect, **G** is the covariance matrix of these random effects, and $\boldsymbol{X}_i$ and $\boldsymbol{Z}_i$ are design matrices for the fixed and random effects, respectively. As such, a DMM may be used to model either linear (in the special case that *f* in Equation 1 is the identity function) or non-linear growth trajectories among learners, but in either case, individual trajectories are estimated for every learner in the model (i.e., $\boldsymbol{b}_i$ carries an *i* subscript, making it person-specific).

Reliability Estimates for Growth Scores

Reliability of DMM growth scores was estimated via Gaussian quadrature over the conditional reliabilities via the following formula[2]:

$$\bar{\rho}_{XX'} = \sum_{k=1}^{200} \varphi^*(b_{1k})(\rho_{XX'} \mid b_{1ik}) \text{ where } \rho_{XX'}\Big| b_{1i} = 1 - \frac{Var(\hat{\beta}_1 + \hat{b}_{1i})}{Var(b_1)} \quad (3)$$

where $\bar{\rho}_{XX'}$ is the approximate marginal reliability, k is an index for the quadrature component, $\varphi^*(b_{1k})$ are normal density weights where $\varphi^*(b_{1k}) = \frac{1}{\sqrt{2\pi}} e^{-.5 b_{1k}^2}$ (standardized so that $\sum \varphi^* = 1$) , and $(\rho_{XX'} \mid b_{1ik})$ is the conditional reliability over the kth component interval. In essence, Equation 3 yields a reliability estimate that represents the proportion of true variance to error variance present in the growth scores.

References

1. Lindstrom MJ, Bates DM. Nonlinear mixed effects models for repeated measures data. Biometrics. 1990;46:673–687.
2. Nicewander WA. Conditional reliability coefficients for test scores. Psychological Methods. 2018;23:351–62.