Counterexample 1. Sign of the bias of unmeasured confounding for non-binary exposures.

Suppose that A is ternary and takes values in the set $\{0, 1, 2\}$ and suppose that a single binary variable U confounds the relationship between A and Y so that the causal relationships are given by the causal directed acyclic graph given in the online eFigure. In this figure, Y depends A and U. We consider a case in which the dependence is deterministic so that $Y = g(a, u)$. Suppose $g(a, u) = a+u-1$ we then have that $g(A = 1, U = 1) = 1$ and $g(A = 1, U = 0) = 0$ then both A and U have a positive average monotonic effect on Y. Suppose further A depends on U and some random term ϵ so that $A = f(u, \epsilon)$ and suppose that $P(U = 0) = P(U = 1) = 1/2$; and that $P(\epsilon = 1) = P(\epsilon = 2) = P(\epsilon = 3) = 1/6$ and $P(\epsilon = 4) = 1/2$; and that $f(u, \epsilon = 1) = 2$ for $u = 0, 1$, $f(u, \epsilon = 2) = 0$ for $u = 0, 1$, $f(U = 1, \epsilon = 3) = 2$, $f(U = 0, \epsilon = 3) = 1$, $f(U = 1, \epsilon = 4) = 1$ and $f(U = 0, \epsilon = 4) = 0$ so that U has a positive average monotonic effect on A. We then have $P(U = 1|A = 1) = P(U = 1, A = 1)/P(A = 1) = (1/2)(1/2)/\{(1/2)(1/2) + (1/6)(1/2)\} = 3/4$ and so $P(U = 0|A = 1) = 1/4$. Thus the true causal effect on Y of intervening to set $A = 1$ will be given by $\sum_u \mathbb{E}[Y|A = 1, U = u]P(U = u) = (1)(1/2) + (0)(1/2) = 1/2$ whereas the estimate without controlling for U will be given by $\mathbb{E}[Y|A = 1] = \sum_u \mathbb{E}[Y|A = 1, U = u]P(U = u|A = 1) = (1)(3/4)+(0)(1/4) = 3/4$. In this case the estimate of the causal effect without controlling for U exceeds the true causal effect on Y of intervening to set $A = 1$. It can also be verified in this case that $(\mathbb{E}[Y|A = 2] - \mathbb{E}[Y|A = 1]) - (\mathbb{E}[Y_2] - \mathbb{E}[Y_1]) = (5/3 - 3/4) - 1 = -1/12 < 0$. If however the probabilities for ϵ had been $P(\epsilon = 1) = P(\epsilon = 2) = P(\epsilon = 4) = 1/6$ and $P(\epsilon = 3) = 1/2$ then we would still have the positive average monotonic effects indicated in the figure but the estimate of without controlling for U would then be $1/4$ which would be less than $1/2$, the true causal effect on Y of intervening to set $A = 1$. It can be verified in this case that $(\mathbb{E}[Y|A = 2] - \mathbb{E}[Y|A = 1]) - (\mathbb{E}[Y_2] - \mathbb{E}[Y_1]) = (9/5 - 1/4) - 1 = 11/20 > 0$. For intermediate values of the intervention variable we thus see that the bias when control for confounding is inadequate may be in either direction even in the presence of positive average monotonic effects.

Counterexample 2. Positive average monotonic effects are insufficient when the condtional independence condition is not met.

Consider the causal directed acyclic graph given in Figure 5 with signs given to the $V - A$, the $V - W$, the $W - A$ and the $W - Y$ edges to represent positive average monotonic effects i.e. to indicate that intervening to increase V increases both A and W on average and intervening to increase W increases A and Y on average. Note that V and W are not independent because V is a cause of W. Suppose in this example that A is binary and that there is no causal effect of A on Y so that there is no $A - Y$ edge in Figure 5. Suppose that $P(V = 1) = P(V = 0) = 1/2$. Suppose also that $W_{v=1} = 1$ and that $P(W_{v=0} = 0) = 2/3$ and $P(W_{v=0} = 2) = 1/3$ so that $\mathbb{E}[W|V = 1] = 1$ and $\mathbb{E}[W|V = 0] = 2/3$. Then V has a positive average monotonic effect on W. Suppose further that $P(A_{v=1,w} = 1) = 2/3$ and $P(A_{v=1,w} = 0) = 1/3$ and that $P(A_{v=0,w} = 1) = 1/3 + W/6$ and $P(A_{v=0,w} = 0) = 2/3 - W/6$ so that $\mathbb{E}[A|V = 1, W] = 2/3$ and $\mathbb{E}[A|V = 0, W] = 1/3 + W/6$. Thus W has a positive average monotonic effect on A. Furthermore, since $W \leq 2$ it also follows that $\mathbb{E}[A|V = 0, W] = 1/3 + W/6 \leq 2/3 = \mathbb{E}[A|V = 1, W]$ and thus V has a positive average monotonic effect on A. Finally suppose that $Y_{w=0} = Y_{w=1} = 0$ and $Y_{w=2} = 1$ so that $\mathbb{E}[Y|W = 0] = \mathbb{E}[Y|W = 1] = 0$ and $\mathbb{E}[Y|W = 2] = 1$ from which it follows that W has a positive average monotonic effect on Y. We thus have the signed edges given in Figure 5. Clearly in this example $\mathbb{E}[Y_{a=1}] - \mathbb{E}[Y_{a=0}] = 0$ since A has no causal effect on Y. However, we can calculate the following probabilities: we have $P(A = 1|V = 1) = P(A = 1|W = 1, V = 1) = 2/3$ and $P(A = 1|V = 0) = P(A = 1|V = 0, W = 0)P(W = 0) + P(A = 1|V = 0, W = 2)P(W = 2) = (1/3)(2/3) + (1/3 + 2/6)(1/3) = 4/9$. From this it follows that $P(A = 1) = P(A = 1|V = 0)P(V = 0) + P(A = 1|V = 1)P(V = 1) = (4/9)(1/2) + (2/3)(1/2) = 5/9$. Also, $P(V = 1|A = 1) = P(A = 1|V = 1)P(V = 1)/P(A = 1) = (2/3)(1/2)/(5/9) = 3/5$ and $P(V = 1|A = 0) = P(A = 1|V = 0)P(V = 0)/P(A = 0) = (4/9)(1/2)/(5/9) = 2/5$. We can then calculate $\mathbb{E}[Y|A = 1] = \sum_v \mathbb{E}[Y|A = 1, V = v]P(V = v|A = 1) = \sum_v P(W = 2|V = v)P(V = v|A = 1) = (1/3)(2/5) + (0)(3/5) = 2/15$ and $\mathbb{E}[Y|A = 0] = \sum_v \mathbb{E}[Y|A = 0, V = v]P(V = v|A = 0) = \sum_v P(W = 2|V = v)P(V = v|A = 0) = (1/3)(3/5) + (0)(2/5) = 3/15$. From this it follows that $\mathbb{E}[Y|A = 1] - \mathbb{E}[Y|A = 0] = -1/15$ is an underestimate of $\mathbb{E}[Y_{a=1}] - \mathbb{E}[Y_{a=0}] = 0$. If, on the other hand we had that $W_{v=1} = 2$

and that $P(W_{v=0} = 0) = 2/3$ and $P(W_{v=0} = 2|V = 0) = 1/3$ so that $\mathbb{E}[W|V = 1] = 2$ and $\mathbb{E}[W|V = 0] = 2/3$ we would again have the signed edges given in Figure 5 but in this case $\mathbb{E}[Y|A = 1] = \sum_v \mathbb{E}[Y|A = 1, V = v]P(V = v|A = 1) = \sum_v P(W = 2|V = v)P(V = v|A = 1) = (1/3)(2/5) + (1)(3/5) = 11/15$ and $\mathbb{E}[Y|A = 0] = \sum_v \mathbb{E}[Y|A = 0, V = v]P(V = v|A = 0) = \sum_v P(W = 2|V = v)P(V = v|A = 0) = (1/3)(3/5) + (1)(2/5) = 9/15$ and thus $\mathbb{E}[Y|A = 1] - \mathbb{E}[Y|A = 0] = 2/15$ would be an overestimate of $\mathbb{E}[Y_{a=1}] - \mathbb{E}[Y_{a=0}] = 0$. We have shown that if the conditional independence condition of Result 1 does not hold then the resultant bias to the causal risk difference can be of either sign.

Counterexample 3. Positive average monotonic effects are not transitive. Consider the causal directed acyclic graph given in Figure 6 with signs given to the $A - B$ and the $B - C$ edges to represent positive average monotonic effects. Suppose that $P(A = 1) = P(A = 0) = 1/2$. Suppose also that if $A = 1$ then $P(B = 1) = 1$ and that if $A = 0$ then $P(B = 0) = 2/3$ and $P(B = 2) = 1/3$ so that $\mathbb{E}[B_{a=1}] = 1$ and $\mathbb{E}[B_{a=0}] = 2/3$. Finally suppose that if $B = 2$ then $C = 1$ and if $B = 0$ or $B = 1$ then $C = 0$. We then have that increasing A increases B on average and increasing B increases C on average but in this example, when $A = 1$ then $B = 1$ and $C = 0$ but when $A = 0$ then $P(B = 0) = 2/3$ and $P(B = 2) = 1/3$ and so $P(C = 0) = 2/3$ and $P(C = 1) = 1/3$. Thus $\mathbb{E}[C_{a=1}] = 0$ but $\mathbb{E}[C_{a=0}] = 1/3$. Thus intervening to increase A decreases C on average.