

Further Refinements to the Organizational Schema for Causal Effects

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eAppendix

Sufficient conditions for no confounding when odds ratios are used

When a reference of exposure is $A = 0$, a causal odds ratio in the total population is given by

$$\frac{P[Y_1 = 1]}{(1 - P[Y_1 = 1])} \bigg/ \frac{P[Y_0 = 1]}{(1 - P[Y_0 = 1])} = \frac{r_1 + r_2}{r_3 + r_4} \bigg/ \frac{r_1 + r_3}{r_2 + r_4}.$$

Likewise, causal odds ratios in the exposed and the unexposed groups are given by

$$\frac{P[Y_1 = 1 | A = 1]}{(1 - P[Y_1 = 1 | A = 1])} \bigg/ \frac{P[Y_0 = 1 | A = 1]}{(1 - P[Y_0 = 1 | A = 1])} = \frac{p_1 + p_2}{p_3 + p_4} \bigg/ \frac{p_1 + p_3}{p_2 + p_4}$$

and

$$\frac{P[Y_1 = 1 | A = 0]}{(1 - P[Y_1 = 1 | A = 0])} \bigg/ \frac{P[Y_0 = 1 | A = 0]}{(1 - P[Y_0 = 1 | A = 0])} = \frac{q_1 + q_2}{q_3 + q_4} \bigg/ \frac{q_1 + q_3}{q_2 + q_4},$$

respectively. Meanwhile, an associational odds ratio is described as

$$\frac{P[Y = 1 | A = 1]}{(1 - P[Y = 1 | A = 1])} \bigg/ \frac{P[Y = 1 | A = 0]}{(1 - P[Y = 1 | A = 0])} = \frac{p_1 + p_2}{p_3 + p_4} \bigg/ \frac{q_1 + q_3}{q_2 + q_4}.$$

When we use the exposed group as the target population, a sufficient condition for no confounding in distribution is given by

$$(p_1 + p_3) = (q_1 + q_3). \quad [\text{Eq. 1}]$$

Meanwhile, a sufficient condition for no confounding in measure is given by

$$\begin{aligned} \frac{p_1 + p_2}{p_3 + p_4} \bigg/ \frac{p_1 + p_3}{p_2 + p_4} &= \frac{p_1 + p_2}{p_3 + p_4} \bigg/ \frac{q_1 + q_3}{q_2 + q_4} \\ \Leftrightarrow (p_1 + p_3) &= (q_1 + q_3), \quad [\text{Eq. 2}] \\ (\Leftrightarrow (p_2 + p_4) &= (q_2 + q_4)) \end{aligned}$$

which is equivalent to Equation 1. Likewise, when we use the unexposed group as the target population, a sufficient condition for no confounding in distribution is given by

$$(p_1 + p_2) = (q_1 + q_2). \quad [\text{Eq. 3}]$$

Meanwhile, a sufficient condition for no confounding in measure is given by

$$\begin{aligned} \frac{q_1 + q_2}{q_3 + q_4} \bigg/ \frac{q_1 + q_3}{q_2 + q_4} &= \frac{p_1 + p_2}{p_3 + p_4} \bigg/ \frac{q_1 + q_3}{q_2 + q_4} \\ \Leftrightarrow (p_1 + p_2) &= (q_1 + q_2), \quad [\text{Eq. 4}] \\ (\Leftrightarrow (p_3 + p_4) &= (q_3 + q_4)) \end{aligned}$$

which is equivalent to Equation 3. Thus, as remarked in the text, Equations 1 to 4 clearly show that, when the target population is the exposed or the unexposed, sufficient conditions for no confounding are identical in the two definitions of confounding.

By contrast, when the total population is the target population, it is crucial to distinguish the notions of confounding in distribution and confounding in measure because sufficient conditions for no confounding vary according to these notions.

As explained in the text, when the total population is the target population, a sufficient condition for no confounding in distribution is given by

$$\{(p_1 + p_3) = (q_1 + q_3)\} \wedge \{(p_1 + p_2) = (q_1 + q_2)\}. \quad [\text{Eq. 5}]$$

Meanwhile, a sufficient condition for no confounding in measure is given by

$$\begin{aligned} & \frac{r_1 + r_2}{r_3 + r_4} \bigg/ \frac{r_1 + r_3}{r_2 + r_4} = \frac{p_1 + p_2}{p_3 + p_4} \bigg/ \frac{q_1 + q_3}{q_2 + q_4} \\ & \Leftrightarrow \frac{(r_1 + r_2)(r_2 + r_4)}{(r_1 + r_3)(r_3 + r_4)} = \frac{(p_1 + p_2)(q_2 + q_4)}{(q_1 + q_3)(p_3 + p_4)} \\ & \Leftrightarrow (r_1 + r_2)(r_2 + r_4)(q_1 + q_3)(p_3 + p_4) = (r_1 + r_3)(r_3 + r_4)(p_1 + p_2)(q_2 + q_4) \\ & \Leftrightarrow \{(p_1 + p_2)P[A=1] + (q_1 + q_2)P[A=0]\} \{(p_2 + p_4)P[A=1] + (q_2 + q_4)P[A=0]\} (q_1 + q_3)(p_3 + p_4) \\ & \quad = \{(p_1 + p_3)P[A=1] + (q_1 + q_3)P[A=0]\} \{(p_3 + p_4)P[A=1] + (q_3 + q_4)P[A=0]\} (p_1 + p_2)(q_2 + q_4) \\ & \Leftrightarrow \{(p_1 + p_2)(p_2 + p_4)(P[A=1])^2 + (p_1 + p_2)(q_2 + q_4)P[A=1]P[A=0] + (p_2 + p_4)(q_1 + q_2)P[A=1]P[A=0] + (q_1 + q_2)(q_2 + q_4)(P[A=0])^2\} (q_1 + q_3)(p_3 + p_4) \\ & \quad = \{(p_1 + p_3)(p_3 + p_4)(P[A=1])^2 + (p_1 + p_3)(q_3 + q_4)P[A=1]P[A=0] + (p_3 + p_4)(q_1 + q_3)P[A=1]P[A=0] + (q_1 + q_3)(q_3 + q_4)(P[A=0])^2\} (p_1 + p_2)(q_2 + q_4) \\ & \Leftrightarrow \{(p_1 + p_2)(p_2 + p_4)(P[A=1])^2 + (p_2 + p_4)(q_1 + q_2)P[A=1]P[A=0] + (q_1 + q_2)(q_2 + q_4)(P[A=0])^2\} (q_1 + q_3)(p_3 + p_4) \\ & \quad = \{(p_1 + p_3)(p_3 + p_4)(P[A=1])^2 + (p_1 + p_3)(q_3 + q_4)P[A=1]P[A=0] + (q_1 + q_3)(q_3 + q_4)(P[A=0])^2\} (p_1 + p_2)(q_2 + q_4). \quad [\text{Eq. 6}] \end{aligned}$$

When the outcome of interest is rare, Equation 6 can be approximately reduced to

$$\begin{aligned}
& \left\{ (p_1 + p_2)(P[A=1])^2 + (q_1 + q_2)P[A=1]P[A=0] + (q_1 + q_2)(P[A=0])^2 \right\} (q_1 + q_3) \\
&= \left\{ (p_1 + p_3)(P[A=1])^2 + (p_1 + p_3)P[A=1]P[A=0] + (q_1 + q_3)(P[A=0])^2 \right\} (p_1 + p_2) \quad (\because \{p_3 + p_4 \approx 1\} \wedge \{p_2 + p_4 \approx 1\} \wedge \{q_3 + q_4 \approx 1\} \wedge \{q_2 + q_4 \approx 1\}) \\
&\Leftrightarrow \left\{ (p_1 + p_2)(P[A=1])^2 + (q_1 + q_2)P[A=0](P[A=1] + P[A=0]) \right\} (q_1 + q_3) = \left\{ (p_1 + p_3)P[A=1](P[A=1] + P[A=0]) + (q_1 + q_3)(P[A=0])^2 \right\} (p_1 + p_2) \\
&\Leftrightarrow \left\{ (p_1 + p_2)(P[A=1])^2 + (q_1 + q_2)P[A=0] \right\} (q_1 + q_3) = \left\{ (p_1 + p_3)P[A=1] + (q_1 + q_3)(P[A=0])^2 \right\} (p_1 + p_2) \quad (\because P[A=1] + P[A=0] = 1) \\
&\Leftrightarrow (p_1 + p_3)(p_1 + p_2)P[A=1] - (p_1 + p_2)(q_1 + q_3)(P[A=1])^2 = (q_1 + q_2)(q_1 + q_3)P[A=0] - (p_1 + p_2)(q_1 + q_3)(P[A=0])^2 \\
&\Leftrightarrow (p_1 + p_2) \left\{ (p_1 + p_3) - (q_1 + q_3)P[A=1] \right\} \times P[A=1] = (q_1 + q_3) \left\{ (q_1 + q_2) - (p_1 + p_2)P[A=0] \right\} \times P[A=0] \\
&\Leftrightarrow (p_1 + p_2) \left\{ (p_1 + p_3) - (q_1 + q_3)(1 - P[A=0]) \right\} \times P[A=1] = (q_1 + q_3) \left\{ (q_1 + q_2) - (p_1 + p_2)(1 - P[A=1]) \right\} \times P[A=0] \quad (\because P[A=1] + P[A=0] = 1) \\
&\Leftrightarrow (p_1 + p_2) \left\{ (p_1 + p_3) - (q_1 + q_3) \right\} \times P[A=1] = (q_1 + q_3) \left\{ (q_1 + q_2) - (p_1 + p_2) \right\} \times P[A=0], \quad [\text{Eq. 7}]
\end{aligned}$$

which is equivalent to the sufficient condition for no confounding in measure for the total population when risk ratios are used (see text).