APPENDIX A. Ocular-Surface Temperature Physical Model

One-dimensional conservation of thermal energy in the composite cornea of Figure 4 reads

$$\frac{\partial T(t,x)}{\partial t} = \alpha \frac{\partial^2 T(t,x)}{\partial x^2} \qquad 0 < x < L \tag{A1}$$

where $T(t,x)$ is the transient absolute temperature in the composite cornea, t is time, x is the distance from the back of the crystalline lens to the tear film, and $\alpha \equiv k / \rho\hat{C}_p$ is the thermal diffusivity of the composite cornea. Effective thermal properties k and $\rho\hat{C}_p$ are estimated from the sum of thermal resistances (i.e., $L / k = \sum_i L_i / k_i$) and volume-averaged heat capacities (i.e., $\rho\hat{C}_p = \sum_i L_i \rho_i \hat{C}_{p,i} / L$) over the composite cornea of thickness *L*. k_i, L_i, ρ_i, and $\hat{C}_{pi}$ are the thermal conductivity, thickness, mass density, and specific heat capacity of ocular material *i* that contributes thermal resistance to the composite cornea (i.e., the crystalline lens, anterior chamber, and cornea). Values of thickness, thermal conductivity, and mass density, and specific heat capacity for each material are listed in Table A1.

Table A1. Thermal resistance of composite cornea[28]*

	L [μm]	k [W/m/K]	ρ [kg/m^3]	$\hat{C}_p$ [J/kg/K]
Crystalline Lens	4000	0.40	1050	3000
Anterior Chamber	3000	0.58	996	3997
Cornea	535	0.58	1050	4178
Overall	7535	0.47	1028	3478

*L, k, ρ, and $\hat{C}_p$ are the thickness, thermal conductivity, mass density, and specific heat capacity of the listed ocular material, respectively.

Solution of Eq. A1 requires an initial condition and two boundary conditions. Upon lid opening, an initial non-uniform temperature profile, $T(0,x)$, exists across the eye, as illustrated in Figure 4. For convenience, we approximate the initial temperature profile as pseudo-steady.

$$T(0,x) = T_o(x) = T_B - \frac{x}{L}[T_B - T_S(0)] \tag{A2}$$

Remaining boundary conditions include isothermal body temperature behind the lens

$$T(t,0) = T_B \tag{A3}$$

and continuity of heat flux at the ocular surface/environment boundary

$$-k\frac{\partial T(t,L)}{\partial x} = \hat{J}_E \Delta\hat{H}_{vap} + h_{eff}[T(t,L) - T_\infty] \tag{A4}$$

where $\hat{J}_E$ is the area-averaged mass evaporation flux of the tear film (i.e., the evaporation rate), $\Delta\hat{H}_{vap}$ is the specific enthalpy of vaporization of water, $T(t,L) \equiv T_S(t)$ is the measured ocular-surface temperature, and T_∞ is environmental temperature. In Eq. A4, h_{eff} is the effective heat transfer coefficient for the parallel processes of natural convection and radiation or $h_{eff} = h_{nat} + h_{rad}$, where h_{nat} and h_{rad} are natural-convective and radiative heat transfer coefficients, respectively. tear-film evaporation rate enters the problem through Eq. A4 since $\hat{J}_E$ = tear-film evaporation rate. Accordingly, the product $\hat{J}_E \Delta\hat{H}_{vap}$ gives the evaporative heat loss from the cornea.

Analytic solution to Eqns. A1 – A4 by separation of variables[46] gives the transient temperature profile in the composite cornea as

$$\theta(\tau,\xi) = \frac{\mathrm{B}(1+E)}{1+B}\xi + \sum_{k=1}^{\infty} A_k \sin\lambda_k \xi \exp(-\lambda_k^2 \tau) \tag{A5}$$

where

$$A_k = \left[\theta_S(0) - \frac{B(1+E)}{1+B}\right] \frac{\int_0^1 \xi \sin(\lambda_k \xi) d\xi}{\int_0^1 \sin^2(\lambda_k \xi) d\xi}, \tag{A6}$$

$$\tan\lambda_k + \lambda_k / B = 0, \tag{A7}$$

and $1-\theta(\tau,\xi) = [T(t,x) - T_\infty]/[\mathrm{T}_B - T_\infty]$ is dimensionless temperature,

$1-\theta_S(0) = [T_S(0) - T_\infty]/[\mathrm{T}_B - T_\infty]$ is dimensionless corneal surface temperature at zero time, $\tau = \alpha t / L^2$ is dimensionless time, $\xi = x/L$ is dimensionless distance, $B = h_{eff} L / k$ is the Biot number[2], $E = \hat{J}_E \Delta\hat{H}_{vap} / h_{eff}\left(T_B - T_\infty\right)$ is an evaporation number defined as the ratio of evaporative heat loss to environmental heat loss, and λ_k is the k^{th} eigenvalue in Eq. A7.[47] Values of λ_k are obtained numerically as discussed below.

Evaluation of the integrals in Eq. A6 and substitution into Eq. A5 with $\xi = 1$ gives the desired dimensionless corneal surface temperature as

$$\theta_S(\tau) = \frac{B[1+E]}{1+B}\left[1 + \frac{2(1+B)^2}{B^2[1+E]}\left(\theta_S(0) - \frac{B[1+E]}{1+B}\right)\sum_{k=1}^{\infty}\left(\frac{\sin\lambda_k \cos\lambda_k}{\sin\lambda_k \cos\lambda_k - \lambda_k}\right)\exp\left(-\lambda_k^2 \tau\right)\right] \tag{A8}$$

Eq. A8 specifies the corneal surface-temperature decline as a combination of exponential decays rather than a single exponential decay.[19]

Eq. A8 strictly holds only when $\hat{J}_E$ is constant. If $\hat{J}_E$ varies in time because of black spot/streak formation and area increase, coupled Eqs. 2 and A8 are solved numerically by Newton iteration in Matlab R2010a (The Math Works Inc., Natick, MA). In all cases, we take the first 100 terms in the summation of Eq. A6. Eigenvalues, λ_k, in Eq. A14 are obtained numerically by Newton iteration and validated against known literature values.[47] Finally, to determine human tear-film evaporation rate through the tear-film lipid-layer, $\beta\hat{J}_W$, Eqs. 2 and A8 are best fit by minimizing least-square errors to measured dynamic ocular-surface temperatures using only independently determined parameters: h_{eff}, $\hat{J}_W$, and a. Briefly, $\hat{J}_W$ and h_{eff} are measured as functions of temperature from separate *in-vitro* water-evaporation experiments (see Appendix B). Then, $\hat{J}_W$ and h_{eff} are held constant in fitting the *in-vivo* ocular-surface temperature data to Eqs. 2 and A8. These two known parameters strictly depend on temperature. However, they are evaluated here at the subject-dependent initial tear surface temperature, $T_S(t=0)$, since typical temperature declines are less than 1 °C (see Figure 3).

Since ocular-surface temperature is many times assumed to decline linearly with time,[15] it is helpful to consider early-time solution to Eqs. A1 – A4. This task is readily accomplished in Laplace space, *s*, expanded for large *s*, and inverted by table[48] to yield

$$\theta_S(\tau)-\theta_S(0)=\frac{2\left[B[1+E]-\theta_S(0)[1+B]\right]}{\sqrt{\pi}}\sqrt{\tau} \qquad \text{(A9)}$$

Thus, ocular-surface temperature initially declines as time to the ½ power, not linearly. This result holds even when tear evaporation rate is constant since at early time $\hat{J}_E=\beta\hat{J}_W$. That is, at early time there is little to no lipid-layer breakup; all evaporation occurs through an intact, fully functioning tear-film lipid-layer.

REFERENCES ONLY APPEARING IN APPENDIX A

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