Modeling the Allocation System: Principles for Robust Design before Restructuring

SDC, APPENDIX: SENSITIVITY TO SMALL DATA CHANGES IN OPTIMIZATION PROBLEMS

The following is a well-known example of how small margins of error can dramatically affect the quality of a solution to an optimization problem [1]. Consider a company that produces two drugs, Drug I and Drug II containing a specific active agent A, which is extracted from two types of raw materials purchased on the market, Raw I and Raw II. Let $x_{Drug\ I}, x_{Drug\ II}, x_{Raw\ I}, x_{Raw\ II}$, be the nonnegative amounts of Drugs I and II produced and quantities of Raw I and Raw II purchased respectively.

The objective is to maximize profit, which is given by the equation: $-5500x_{Drug\ I} - 6100x_{DrugI\ I} + 100x_{Raw\ I} + 199.9x_{RawI\ I}$.This profit maximization is subject to the following constraints:

$$0.05x_{Drug\ I} + 0.6x_{DrugI\ I} - 0.01x_{Raw\ I} - 0.02x_{RawI\ I} \leq 0$$

$$x_{Raw\ I} + x_{RawI\ I} \leq 1000$$

$$90x_{Drug\ I} + 100x_{DrugI\ I} \leq 2000$$

$$40x_{Drug\ I} + 50x_{DrugI\ I} \leq 800$$

$$700x_{Drug\ I} + 800x_{DrugI\ I} + 100x_{Raw\ I} + 199.9x_{RawI\ I} \leq 10000$$

These constraints describe requirements such as only being able to store 1,000 units of raw materials. The details regarding the constraints are not important in themselves except for the fact that they correspond to realistic constraints that a decision maker may face. The optimization problem can be solved to find the best choices of Drug I and Drug II to produce and the quantities of raw materials to purchase. The optimal profit after solving the above problem is $8,820.

Unfortunately, the solution to the above problem assumes that all the coefficients are known exactly and accurately. Suppose the first constraint is replaced with the following:

$$0.05x_{Drug\ I} + 0.6x_{DrugI\ I} - 0.01x_{Raw\ I} - 0.0196x_{RawI\ I} \leq 0$$

This new constraint is the same as before, except that the coefficient for $x_{RawI\ I}$ has been changed by only 2%. Solving the problem again with this constraint in lieu of the original and all other constraints as before, yields an optimal profit of $6,929 – thus, a small 2% perturbation in one coefficient of the model led to a 21% decrease in profit!

This example demonstrates the potential severity of minor errors and reliance on estimated data in solving deterministic optimization problems. Such applications of mathematical programming techniques require further sensitivity analyses or the use of specialized techniques for safer and more reliable implementation in practice.

REFERENCE

1. Ben-Tal A, El Ghaoui L, and Nemirovski A. Robust optimization. Princeton University Press, 2009.