eAppendix

Detailed mathematical derivation:

Let E, U, M and D be four random variables with

$$\begin{tikzcd} E \arrow[r] & U \arrow[r] \arrow[d] & D \\ & M & \end{tikzcd}$$

Suppose the linear models among them are:

$$U = \alpha_U + \beta_U \; E + \varepsilon_U$$

$$M = \alpha_M + \beta_M \; U + \varepsilon_M$$

$$D = \alpha_D + \beta_D \; U + \varepsilon_D$$

Assume we have E ~ N (μ_E, σ_E^2), ε_U ~ N (0, σ_U^2), ε_M ~ N (0, σ_M^2) and ε_D ~ N (0, σ_D^2), where

μ_E, σ_E^2, σ_U^2, σ_M^2, and σ_D^2 are known.

$$\mathrm{Var}(U) = \beta^2{}_U \sigma^2{}_E + \sigma_U^2,$$

$$\mathrm{Var}(M) = \beta^2{}_M \sigma^2{}_U + \epsilon^2{}_M = \beta^2{}_M \beta^2{}_U \sigma^2{}_E + \beta^2{}_M \epsilon^2{}_U + \sigma_M^2,$$

$$\mathrm{Var}(D) = \beta^2{}_D \sigma^2{}_U + \epsilon^2{}_D = \beta^2{}_D \beta^2{}_U \sigma^2{}_D + \beta^2{}_D \epsilon^2{}_U + \sigma_D^2.$$

Correlation between E and M

Therefore

$$\begin{aligned} M &= \alpha_M + \beta_M(\alpha_U + \beta_U E + \varepsilon_U) + \varepsilon_M \\ &= \alpha_M + \beta_M \alpha_U + \beta_M \beta_U E + \beta_M \varepsilon_U + \varepsilon_M \\ &= \alpha'_M + \beta'_M E + \varepsilon'_M, \qquad (1) \end{aligned}$$

where

$$\alpha'_M = \alpha_M + \beta_M \alpha_U,$$

$$\beta'_M = \beta_M \beta_U,$$

$$\varepsilon'_M = \beta_M \varepsilon_U + \varepsilon_M.$$

Similarly, since

$$U = -\frac{\alpha_D}{\beta_D} + \frac{1}{\beta_D} D - \frac{\varepsilon_D}{\beta_D},$$

$$M = \alpha_M + \beta_M\left(-\frac{\alpha_D}{\beta_D} + \frac{1}{\beta_D} D - \frac{\varepsilon_D}{\beta_D}\right) + \varepsilon_M$$

$$= \alpha_M + \beta_M\left(-\frac{\alpha_D}{\beta_D}\right) + \frac{\beta_M}{\beta_D} D - \frac{\beta_M}{\beta_D}\varepsilon_D + \varepsilon_M$$

$$= \alpha''_M + \beta''_M D + \varepsilon''_M, \qquad (2)$$

where

$$\alpha''_M = \alpha_M - \frac{\alpha_D \beta_M}{\beta_D},$$

$$\beta''_M = \frac{\beta_M}{\beta_D},$$

$$\varepsilon''_M = \varepsilon_M - \frac{\beta_M}{\beta_D}\varepsilon_D.$$

Therefore, we have

$$\mathrm{cov}\,(M,E) = \beta_M \beta_U \sigma_E \sigma_M,$$

$$\mathrm{cov}\,(M,U) = \beta_M \sigma_U \sigma_M,$$

$$\mathrm{cov}\,(M,D) = \frac{\beta_M}{\beta_D} \sigma_D \sigma_M.$$

Correlation between E and D

Consider the correlation between E and D.

$$D = \alpha_D + \beta_D(\alpha_U + \beta_U E + \varepsilon_U) + \varepsilon_D$$

$$= \alpha_D + \beta_D\alpha_U + \beta_D\beta_U E + \beta_D\varepsilon_U + \varepsilon_D$$

$$= \alpha'_D + \beta'_D E + \varepsilon'_D, \qquad (3)$$

where

$$\alpha'_D = \alpha_D + \beta_D\alpha_U,$$

$$\beta'_D = \beta_D\beta_U,$$

$$\varepsilon'_D = \beta_D\varepsilon_U + \varepsilon_D.$$

$$D = \alpha_D + \beta_D\left(-\frac{\alpha_M}{\beta_M} + \frac{1}{\beta_M} M - \frac{\varepsilon_M}{\beta_M}\right) + \varepsilon_D$$

$$= \alpha_D + \beta_D\left(-\frac{\alpha_M}{\beta_M}\right) + \frac{\beta_D}{\beta_M} M - \frac{\beta_D}{\beta_M}\varepsilon_M + \varepsilon_D$$

$$= \alpha''_D + \beta''_D M + \varepsilon''_D, \qquad (4)$$

where

$$\alpha''_D = \alpha_D - \frac{\alpha_M \beta_D}{\beta_M},$$

$$\beta''_D = \frac{\beta_D}{\beta_M},$$

$$\varepsilon''_D = \varepsilon_D - \frac{\beta_D}{\beta_M}\varepsilon_M.$$

Therefore, we have

$$\mathrm{cov}(D, E) = \beta_M \beta_U \sigma_E \sigma_D,$$

$$\mathrm{cov}(D, U) = \beta_M \sigma_U \sigma_D,$$

$$\mathrm{cov}(D, M) = \frac{\beta_M}{\beta_D}\sigma_M \sigma_D.$$

Conditional covariance of E and D given M

$$\mathrm{cor}(E, D) = \beta'_D$$

$$= \beta_D \beta_U$$

$$\mathrm{cor}(E, D|\,M) = \frac{\mathrm{cov}(E, D|M)}{\sqrt{\mathrm{var}(E|M)}\sqrt{\mathrm{var}(D|M)}}$$

$$\mathrm{cov}(E, D|\,M) = E[ED|M] - E[E|M]E[D|M]$$

According to section 1 and 2

$$M = \alpha'_M + \beta'_M E + \varepsilon'_M$$

$$E = -\frac{\alpha'_M}{\beta'_M} + \frac{M}{\beta'_M} + \frac{\varepsilon'_M}{\beta'_M}$$

$$D = \alpha''_D + \beta''_D M + \varepsilon''_D,$$

$$E[E|M] = -\frac{\alpha_M}{\beta_M} + \frac{1}{\beta'_M}M$$

$$E[D|M] = \alpha''_D + \beta''_D M$$

$$E[ED|M] = E[(-\frac{\alpha'_M}{\beta_M} + \frac{M}{\beta'_M} + \frac{\varepsilon'_M}{\beta'_M})(\alpha''_D + \beta''_D M + \varepsilon''_D)|M]$$

Since $E[\varepsilon'_M \mid M] = 0$ and $E[\varepsilon''_D \mid M] = 0$,

$E[E \mid M] * E[D \mid M] = -\frac{\alpha'_M \alpha''_D}{\beta_M} - \frac{\alpha'_M \beta''_D M}{\beta_M} + \frac{M\alpha''_D}{\beta'_M} + \frac{M^2 \beta''_D}{\beta'_M} \mid M$, we have

$$E[ED|M] - E[E|M]\,E[D|M] = E[-\frac{\varepsilon'_M}{\beta'_M}\varepsilon''_D\,|M]$$

$$= E[(-\frac{\beta_M \varepsilon_U + \varepsilon_M}{\beta_M \beta_U})(\varepsilon_D - \frac{\beta_M}{\beta_D} \varepsilon_M) |M]$$

$$= E[\frac{\varepsilon_M^2}{\beta_D \beta_U} |M]$$

$$= \frac{\sigma_M^2}{\beta_D \beta_U}$$

$$\mathrm{var}(E|M) = \frac{1}{(\beta_M^{'})^2} \mathrm{var}(\varepsilon'_M)$$

$$= \frac{\beta_M^2 \sigma_U^2 + \sigma_M^2}{\beta_M^2 \beta_U^2}$$

$$\mathrm{var}(D|M) = \mathrm{var}(\varepsilon_D^{''})$$

$$= \epsilon^2_D + \frac{\beta_D^2}{\beta_M^2} \sigma_M^2$$

Therefore

$$\text{bias} = \mathrm{cor}(E, D| M) - \mathrm{cor}(E, D) = \frac{\frac{\sigma_M^2}{\beta_U \beta_D}}{\sqrt{\frac{\beta_M^2 \sigma_U^2 + \sigma_M^2}{\beta_M^2 \beta_U^2}} \sqrt{\sigma_D^2 + \frac{\alpha_M^2 \beta_D^2}{\beta_M^2} \sigma_M^2}} - \beta_D \beta_U \qquad (5)$$

The second model

Let E, U, M and D be four random variables with.

$$\begin{tikzcd}
E \arrow[r] & U \arrow[r] & D \\
& M \arrow[u] &
\end{tikzcd}$$

Suppose the linear models among them are:

$$U = \alpha_U + \beta_{UE} E + \beta_{UM} M + \varepsilon_U,$$

$$D = \alpha_D + \beta_D U + \varepsilon_D,$$

Assume we have $E \sim N(\mu_E, \sigma_E^2)$, $M \sim N(\mu_M, \sigma_M^2)$, $\varepsilon_U \sim N(0, \sigma_U^2)$, $\varepsilon_M \sim N(0, \sigma_M^2)$ and $\varepsilon_D \sim N(0, \sigma_D^2)$, where μ_E, σ_E^2, μ_M, σ_M^2, σ_U^2, and σ_D^2 are known.

$$\mu_U = \alpha_U + \beta_{UE} \mu_E + \beta_{UM} \mu_M,$$

$$\mu_D = \alpha_D + \beta_D \mu_U$$

$$\mathrm{Var}(U) = \beta^2_{UE} \sigma^2_E + \beta^2_{UM} \sigma^2_M + \sigma_U^2,$$

$$\mathrm{Var}(D) = \beta^2_D \sigma^2_U + \sigma_D^2.$$

Therefore

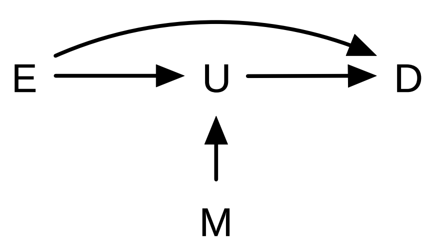
$$
\begin{aligned}
D &= \alpha_D + \beta_D(\alpha_U + \beta_{UE}E + \beta_{UM}M + \varepsilon_U) \\
&= \alpha'_D + \beta'_{DE}E + \beta'_{DM}M + \varepsilon'_D,
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha'_D &= \alpha_D + \beta_D\alpha_U, \\
\beta'_{DE} &= \beta_D\beta_{UE}, \\
\beta'_{DM} &= \beta_D\beta_{UM}, \\
\varepsilon'_D &= \beta_D\varepsilon_U + \varepsilon_D.
\end{aligned}
$$

The third model

Let E, U, M and D be four random variables with



Suppose the linear models among them are:

$$
\begin{aligned}
U &= \alpha_U + \beta_{UE}E + \beta_{UM}M + \varepsilon_U, \\
D &= \alpha_D + \beta_{DE}E + \beta_{DU}U + \varepsilon_D.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
D &= \alpha_D + \beta_{DU}(\alpha_U + \beta_{UE}E + \beta_{UM}M + E_U) + \beta_{DE}E + \varepsilon_D \\
&= \alpha_D + \beta_{DU}\alpha_U + (\beta_{DE} + \beta_{UE}\beta_{DU})E + \beta_{UM}\beta_{DU}M + \beta_{DU}\varepsilon_U + \varepsilon_D \\
&= \alpha'_D + \beta'_{DE}E + \beta'_{DM}M + \varepsilon'_D,
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha'_D &= \alpha_D + \beta_D\alpha_U, \\
\beta'_{DE} &= \beta_{DE} + \beta_{UE}\beta_{DU}, \\
\beta'_{DM} &= \beta_{UM}\beta_{DU}, \\
\varepsilon'_D &= \beta_{DM}\varepsilon_U + \varepsilon_D.
\end{aligned}
$$