High frequency components of auditory evoked potentials are detected in responsive, but not in unconscious patients

Web Appendix: Wavelet Transform

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1 Introduction

For the interested reader we supply more information about wavelet analysis related to our article in anesthesiology. The amount of literature has increased throughout the past 10 years, therefore we apologize for maybe missing out some important articles. Nevertheless we hope that the reader not interested in mathematical formula will still profit by reading the text. Help and new development around wavelet transform can be gathered at Wim Swelden's Wavelet Digest (http://www.wavelet.org/). For a more thorough explanation we refer the reader to the following reviews and articles [1–19].

2 Wavelet transform: basics and explanations

2.1 Overview

Wavelets are a special class of functions which are often used for the analysis of time series and signals. Like with Fourier analysis, which represents a time series as a linear combination of complex exponential functions

$$x(t) = \int_{-\infty}^{+\infty} X(\omega) e^{i2\pi\omega t} d\omega, \tag{1}$$

the basic idea of the wavelet analysis is to represent a time series as a linear combination of wavelets (small locally restricted wave packets). In the Fourier analysis each basis function is associated to a certain frequency f. The absolut values of the resulting coefficients allow the deduction which frequencies contribute to what extend to the signal

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-i2\pi\omega t} dt. \tag{2}$$

Here $X(\omega)$ names complex coefficients which show that relative contribution of the frequencies ω. Equation (2) is the same as the inner product of the signal $x(t)$ with the complex exponential function $e^{-i2\pi\omega t}$ and describes the continuous Fourier transform of the signal $x(t)$. $X(\omega)$ does not provide time information. Therefore, with Fourier analysis we can only basically analyse which frequencies are contained in a signal and to what extent, but not where in the signal.

Opposed to that, the wavelet analysis works with wavelets that are associated with two independent variables, the variable s and the scale τ. In general, each wavelet is equal to zero exept on the intervall $[s-\tau, s+\tau]$. Within this intervall, the mean of the wavelet is zero, the wavelet is centered around s with the length 2τ.

By representing a time series as a linear combination of wavelets, it is possible to show the time evolution of the signals on different scales. We will show this in the following for the continuous and the discrete wavelet transform.

2.2 The continuous wavelet transform

Mathematically strictly formulated a function $\Psi(t)$ is called a wavelet, if it follows the following conditions:

(1) $\int_{-\infty}^{+\infty} \Psi(t)dt = 0$ (mean is zero),
(2) $\int_{-\infty}^{+\infty} \Psi^2(t)dt = 1$ (size is normed to 1).

The properties of such a wavelet in the time-frequency space are described as follows:

$$t_0 = \int_{-\infty}^{+\infty} t|\Psi(t)|^2 dt \quad ; \quad \omega_0 = 2\pi \int_{-\infty}^{+\infty} \omega|\hat{\Psi}(\omega)|^2 d\omega$$

$$D = \sqrt{\int_{-\infty}^{+\infty} (t-t_0)^2 |\Psi(t)|^2 dt}$$

$$B = \sqrt{4\pi^2 \int_{-\infty}^{+\infty} (\omega-\omega_0)^2 |\hat{\Psi}(\omega)|^2 d\omega}$$

$$\mu = D^2 B^2$$

with

$$\int_{-\infty}^{+\infty} |\Psi(t)|^2 dt = 1 \quad \text{and} \quad \hat{\Psi}(\omega) = \int_{-\infty}^{+\infty} \Psi(t) \cdot e^{-i\omega t} dt.$$

The wavelet Ψ is localized around (t_0, ω_0) with the uncertainty μ with the duration D and the bandwidth B. A sharp resolution in the time space and the frequency space together is impossible because of the uncertainty principle [20].

The wavelet transform makes use of translations and dilations of the function $\Psi \in L^2(\mathbf{R})$. In the case of the continuous wavelet transform the translations and dilations vary continuously. The transform therefore uses functions of the following kind:

$$\Psi_{\tau;s}(t) = \frac{1}{\sqrt{|\tau|}}\Psi(\frac{t-s}{\tau}) \quad \text{with} \quad \tau, s \in \mathbf{R},\ \tau \neq 0. \tag{3}$$

These functions are normed which means that the $L^2(\mathbf{R})$ are independent of a. The continuous wavelet transform of a function $f \in L^2(\mathbf{R})$ is defined as follows

$$W(\tau, s) = \langle f, \Psi_{\tau,s} \rangle\ . \tag{4}$$

Mathematically this can be formulated as (Parsevals's identity)

$$2\pi W(\tau, s) = \langle \hat{f}, \hat{\Psi}_{\tau,s} \rangle \tag{5}$$

with

$$\hat{\Psi}_{\tau,s}(\omega) = \frac{\tau}{\sqrt{|\tau|}} e^{-i\omega s}\hat{\Psi}(\tau\omega). \tag{6}$$

If the wavelet ψ follows the condition

$$C_\psi = \int\limits_{-\infty}^{+\infty} \frac{|\hat{\psi}(\omega)|^2}{\omega} d\omega < \infty, \tag{7}$$

then the continuous wavelet transform is invertible $CWT(\tau, s)$ in its defined space and the inverse transform is

$$f(t) = \frac{1}{C_\psi} \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} CWT(\tau, s)\psi_{\tau;s}(t)\frac{d\tau ds}{\tau^2}. \tag{8}$$

A very important relation is described as follows:

$$\int\limits_{-\infty}^{+\infty} f^2(t)dt = \frac{1}{C_\Psi} \int\limits_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} [W(\tau, s)]^2 ds \frac{d\tau}{\tau^2}. \tag{9}$$

The left side of equation 9 is called the "energy", of the signal $x(t)$, in fact it is only an energy in the physical sense when the dimension of $x(t)$ is equivalent.

$[W(\tau,t)]^2/\tau^2$ can be regarded as an energy density function which deconvolutes the energy of $x(t)$ on several time points and scales.

We show an example of the continuous wavelet transform. Figure 1 shows two wavelets. The wavelet on the left side is the Haar Wavelet, which was published as first wavelet in the year 1910 in an article of Haar [21].

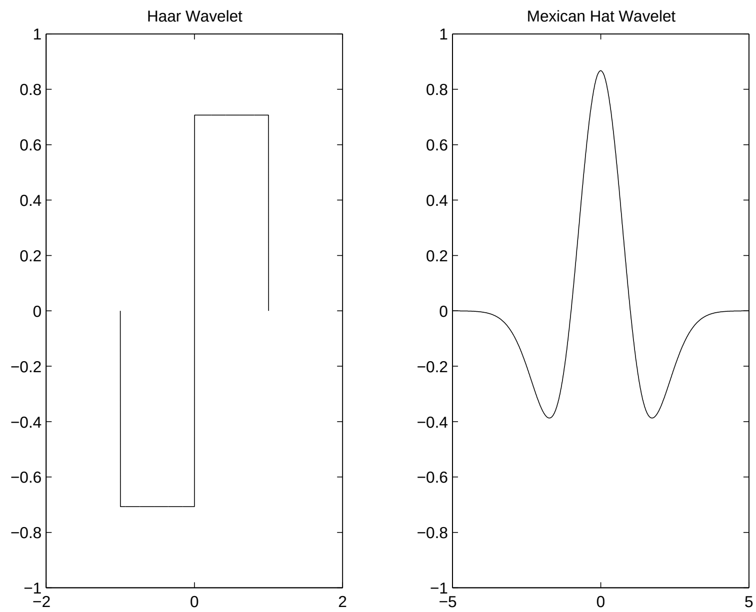

Fig. 1. Two examples of wavelets, on the left side the Haar wavelet, on the right side the Mexican Hat wavelet.

The definition of the Haar wavelet is somehow trivial:

$$\Psi^{(Haar)}(t) = \begin{cases} -1/\sqrt{2}, & n-1 < t < 0 \\ 1/\sqrt{2}, & 0 < t < 1 \\ 0, & \text{otherwise.} \end{cases}$$

The second wavelet $\Psi^{(Mexhat)}(t)$ is proportional to the second derivative of the Gauss' probability function, the naming is derived from the shape of the wavelet.

The continuous wavelet transform of the signal with the Haar wavelet is defined as follows:

$$W^{(Haar)}(\tau, t) = \int_{-\infty}^{+\infty} \Psi_{\tau,s}^{(Haar)}(t) f(t) dt, \begin{cases} \text{with } 0 < \tau < \infty \\ \text{and } \ -\infty < s < +\infty. \end{cases}$$

Conceptionally $\Psi_{\tau,s}^{(Haar)}(t)$ arises from $\Psi^{(Haar)}(t)$ by dilation in such a way that the area different from zero extends to the intervall $[-\tau, \tau]$ followed by a translation so that the function is centered around the time point s. The value, associated with (τ, s) can be interpreted as difference between neighbouring means which are set before and after the time point s on the scale τ.

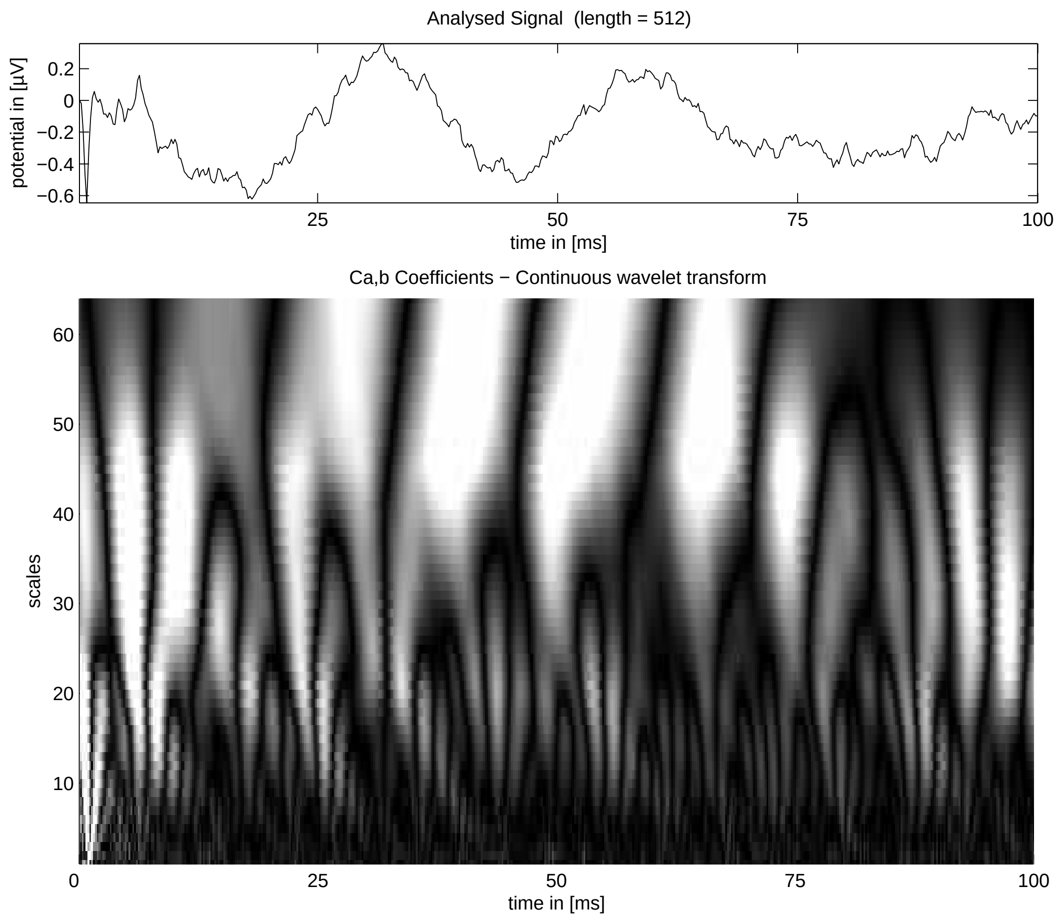

Fig. 2. Continuous wavelet transform of an auditory evoked potential sampled at 5120 Hz with the Daubechies 4 Wavelet on 64 scales.

This description is valid for basically all wavelets. The physical interpretation for the wavelets of a "rounder" shape like the Mexican Hat wavelet, as we presented in Figure 1 results from a comparison of the two wavelets shown in Figure 1: the Haar wavelet calculates simple means of a signal before and after the time point $t = 0$, the Mexican Hat wavelet calculates a weighted mean on an intervall centered around $t = 0$ and weighted means before and after this intervall.

In practice it is of special interest to perform transforms that do not use the complete values of the intervall τ and s for the continuous wavelet transform $CWT(\tau, s)$. The frame theory allows to use only discrete values for τ and s[22]. The most common choice is a dyadic grid, i.e. $\tau = 2^{-j}$ and $s/\tau = l$ with $j, l \in \mathbf{Z}$ [23,24]. This leads to the discrete wavelet transform and their use as a multi resolution analysis. In general we can say that the less values are used for τ and s the more restrictive are the conditions the wavelet has to fulfill. The continuous wavelet transform allows the usage of basically any wavelet.

For many signals the continuous wavelet transform is a highly redundant procedure for both time information and scale information. There will be just a small difference between $W(\tau, s)$ and $W(\tau', s)$, if $|\tau - \tau'|$ is small compared to τ or between $W(\tau, s)$ and $W(\tau, s')$, if $|s - s'|$ is small compared to s. The discrete wavelet transform folds a digital filter with the signal to analyse. The choice of the discrete steps for the translations in time and the dilations to scales τ is principally the choice of the investigator. Nevertheless the complete representation of the signal on different scales without redundancy is only possible for special classes of discrete wavelet transforms. These special classes define an orthogonal deconvololution of the signal on subspaces (which are related to the scales). This leads to the concept of a multi resolution analysis. In the following, we will describe the wavelet transform for the special case that a dyadic grid is chosen for the steps of translation and dilation.

The discrete orthogonal wavelet transform is defined via a wavelet filter and an associated filter, the so called scaling filter. Formally, a wavelet filter $h_{1,l}$ is a sequence of numbers with their sum adding up to zero, the sum of their squares being normable and being orthogonal to translations by the factor of 2:

$$\sum_{l=-\infty}^{\infty} h_{1,l} = 0$$

and

$$\sum_{l=-\infty}^{\infty} h_{1,l} h_{1,l+2n} = \begin{cases} 1 \text{ if } n = 0 \\ 0 \text{ otherwise} \end{cases}$$

The simplest wavelet filters have a finite length L, such that there exists an L with $h_{1,l} = 0$ for $l < 0$ and $l \geq L$, while $h_{1,0} \neq 0$ and $h_{1,L-1} \neq 0$. The Haar wavelet is the simplest wavelet filter:

$$h_{1,l}^{(Haar)} = \begin{cases} -1/\sqrt{2}, \text{ for} & l = 0 \\ 1/\sqrt{2}, \quad \text{for} & l = 1 \\ 0, \qquad \text{otherwise} & \end{cases}$$

The discrete wavelet transform to a scale is defined as follows

$$\tilde{W}_{1,t} = \frac{1}{\sqrt{2}} \sum_{t=0}^{L-1} h_{1,l} X_{t-l mod N} \text{ and}$$

$$\tilde{V}_{1,t} = \frac{1}{\sqrt{2}} \sum_{t=0}^{L-1} g_{1,l} X_{t-lmodN} \quad t = 0, \ldots, N-1.$$

"$t - l\ mod\ N$" is equivalent to "$t - l\ modulo\ N$" and is defined, that if $0 \leq m \leq N-1$ then $m\ mod\ N = m$, otherwise $m\ mod\ N = m + nN$.

The discrete wavelet transform is defined for more than one scale. The practical implementation works by "stretching" the wavelet and the scaling filter in such a way, that their effective length doubles from scale to scale. The length of the filter on scale j then results in $L_j = ((2^j - 1)(L-1) + 1)$.

Today only the filters $h_{1,l}$ und $g_{1,l}$ are used to form a pyramidal algorithm. By definition the scaling coefficients on the scale 0 are equal to the signal $V_{0,t} = X_t$. Once the scaling coefficients $V_{j-1,t}$ on the scale $j-1$ are known, the filtering of that scale with the filters $h_{1,l}$ und $g_{1,l}$ results in the wavelet and scaling coefficients of scale j

$$W_{j,t} = \sum_{l=0}^{L-1} h_{1,l} V_{j-1,2t+1-lmodN_{j-1}} \text{ and}$$

$$V_{j,t} = \sum_{l=0}^{L-1} g_{1,l} V_{j-1,2t+1-lmodN_{j-1}} \text{ for } t = 0, \ldots, N_j - 1.$$

This algorithm allows the wavelet transform to be calculated with $O(N)$ operations. In comparison the algorithm of the fast fourier transform yields $O(Nlog_2(N))$ operations.

Since the discrete wavelet transform describes the deconvolution of a signal on to orthogonal subspaces, the scale J of the discrete wavelet transform can be rewritten as an orthogonal transform of the vector $\mathbf{X} = [X_0, X_1, \ldots, X_{N-1}]'$. If $\mathbf{W}_j = [W_{j,0}, W_{j,1}, \ldots, W_{N_j-1,0}]'$ and $\mathbf{V} = [V_{J,0}, V_{J,1}, \ldots, V_{N_J-1,0}]'$ name the corresponding subspaces, then $\mathbf{W} = W\mathbf{X}$ results in the matrix containing the coefficients of the discrete wavelet transform:

$$\mathbf{W} = \begin{bmatrix} \mathbf{W_1} \\ \mathbf{W_2} \\ \vdots \\ \mathbf{W_J} \\ \mathbf{V_J} \end{bmatrix}. \tag{10}$$

Here W is a $N \times N$ matrix, whose rows depend only on the wavelet filter $h_{1,l}$. The properties of the wavelet filter imply, that W is an orthonormal matrix with $W' \, W = I_N$, where I_N is the $N \times N$ unitary matrix.

The orthogonality yields two important consequences. First, an orthogonal transform conserves physical properties like the "energy" of $\mathbf{X}$ in the sense that $||\mathbf{W}||^2 = ||\mathbf{X}||^2$, where $||\mathbf{X}||^2 = \sum_{t=0}^{N-1} X_t^2$ is the squared norm of the vector $\mathbf{X}$. If we split $\mathbf{W}$ according to equation 10

$$||\mathbf{X}||^2 = \sum_{j=1}^{J} ||\mathbf{W}_j||^2 + ||\mathbf{V}_J||^2. \tag{11}$$

then equation 11 corresponds to the representation of an energy density on different scales. $||\mathbf{W}_j||^2$ corresponds to the contribution of energy that results from changes on the scale τ_j.

Second, the inverse matrix of the matrix containing the coefficients of the discrete wavelet transform W results in the transposed matrix W'. Thus the signal $\mathbf{X}$ can be completely reconstructed via the wavelet coefficients of the discrete wavelet transform $\mathbf{X} = W' \, \mathbf{W}$. $\mathbf{X}$ and $\mathbf{W}$ are therefore equivalent and are both descriptions of the same mathematical identity.

Similar to the deconvolution of $\mathbf{W}$ into the vectors $\mathbf{W}_j$ and $\mathbf{V}_J$ the matrix W can be deconvoluted in

$$W = \begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_J \\ V_J \end{bmatrix}.$$

Here W_j denotes a $N_j \times N$ matrix, with the rows being constructed by the filter $h_{1,l}$. V_J is a $N_J \times N$ matrix calculated via the filter $g_{J,l}$. The signal $\mathbf{X}$ can therefore be described as:

$$\mathbf{X} = W' \, \mathbf{W} = \sum_{j=1}^{J} W_j' \, \mathbf{W}_j + V_J' \, \mathbf{V}_J = \sum_{j=1}^{J} D_j + S_J. \tag{12}$$

$D_j = {W_j}'\mathbf{W}_j$ is a N-dimensional vector, called the j^{th} detail scale, $S_J = {V_J}'\mathbf{V}_J$ describes the approximations of the signal on the scale J.

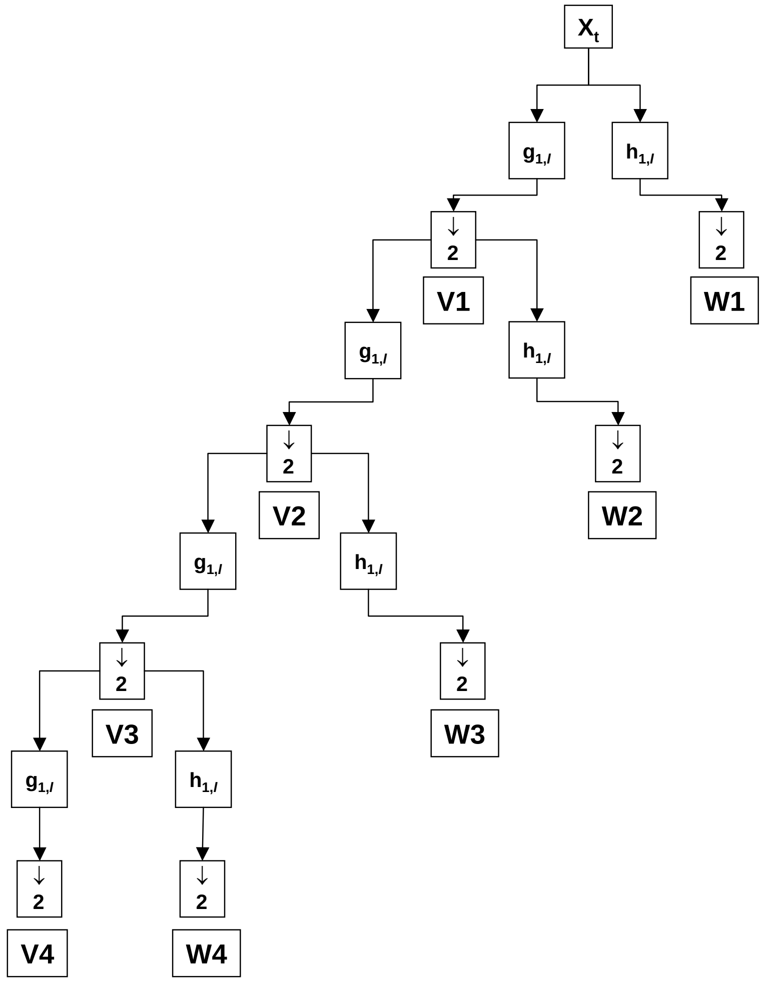

Fig. 3. The successive filtering with the wavelet and the scaling filter, shown as a pyramidal filter bank for a multi resolution analysis down to scale 4.

The described additive deconvolution of the signal $\mathbf{X}$ in detail functions D_j and one approximation function S_J is known as multi resolution analysis.

3 Interindividual replicability

We show the digitally filtered traces of the 19 patients included in our study. The filtering was performed via waveletfiltering and reconstruction as discussed on the previous pages with the wavelet Daubechies 4.

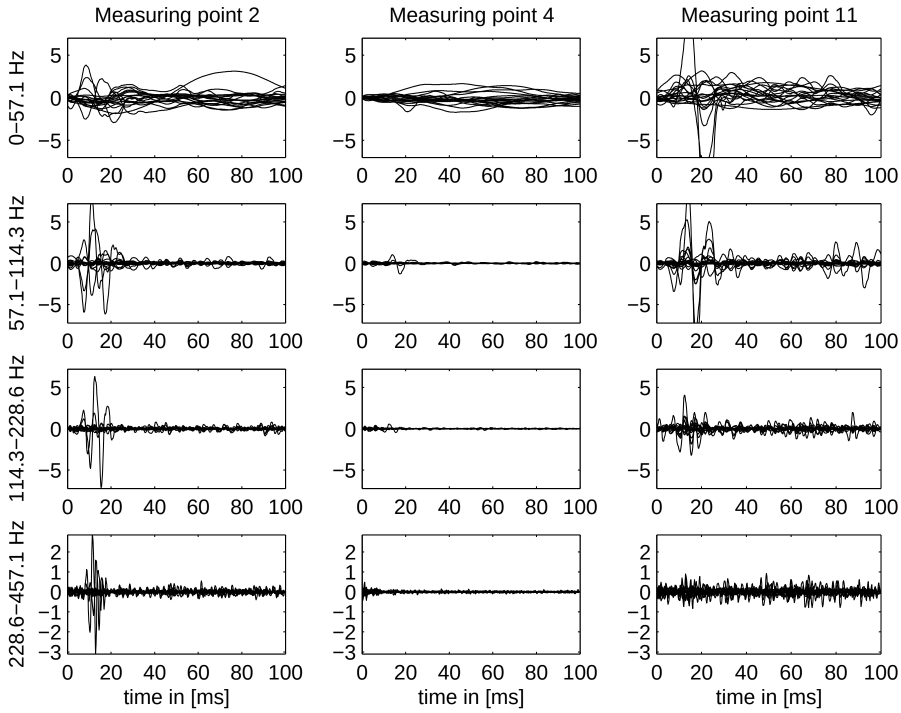

Fig. 4. Digitally filtered traces of the AEPs of 19 patients included in our study. Filtering and reconstruction was performed with the Daubechies 4 wavelet.

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