Supplemental digital content 1

Title: Modeling the recovery of *W′* in the moderate to heavy exercise intensity domain.

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This documents presents the derivation of the mathematical solutions for the different W'_{bal} models presented by Skiba and colleagues (1–4)

W'_{bal} model with only the integrand (1):

$$W'_{bal} = W' - \int_0^t W'_{\exp} \cdot e^{-\left(\frac{(t-u)}{\tau_{W'}}\right)} \qquad [1]$$

W'_{bal} model with "*du*" as the differential variable (2,3):

$$W'_{bal} = W' - \int_0^t W'_{\exp} \cdot e^{-\left(\frac{(t-u)}{\tau_{W'}}\right)} du \qquad [2]$$

W'_{bal} model with "*dt*" as the differential variable (4):

$$W'_{bal} = W' - \int_0^t W'_{\exp} \cdot e^{-\left(\frac{(t-u)}{\tau_{W'}}\right)} dt \qquad [3]$$

In all the forms, W'_{bal} is the *W′* balance at any time during exercise, W'_{exp} is the amount of *W′* expended, (t − u) is the duration of the recovery interval, and $\tau_{W'}$ is the time constant of reconstitution of *W′* in seconds given by,

$$\tau_{W'} = 546 \cdot e^{(-0.01 D_{CP})} + 316 \qquad [4]$$

where, D_{CP} is the difference between CP and average power output during all intervals below CP (recovery power). Equation 2 is a non-linear regression obtained by plotting $\tau_{W'}$ values (calculated by setting the $W'_{bal} = 0$ in Equation 1 at the termination of exercise) against respective D_{CP}s.

Biconditional W'_{bal} model proposed by Skiba and colleagues (2):

$$\begin{aligned} &If\ \ P \geq CP, \quad W'_{bal} = W'_0 - \left[(P - CP) \cdot t\right] \\ &If\ \ P < CP, \quad W'_{bal} = W'_0 - W'_{\exp} \cdot e^{\left(\frac{-D_{CP} \cdot t}{W'_0}\right)} \end{aligned} \qquad [5]$$

where, W'_0 is *W′* at time t=0. The P > CP condition is same as the 2-parameter model for expenditure of *W′*. The P < CP condition models the recovery of *W′*.

Equation 1 cannot be integrated due to the absence of the differential term. Hence, in the following section, Equations 2 and 3 will be integrated to show the difference in the obtained solution, and dimensional analyses will be conducted for both solutions to show the imbalance of units. Additionally, a detailed derivation of the P < CP condition of Equation 5 will be presented.

Note: The constant of integration does not appear in the solutions as the limits of integration are known for all the integrals (i.e. they are considered to be definite integrals).

Integration of Equations 2 and 3

In both the following integrations, the W'_{exp} term is treated as a constant.

Integration of Equation 2

Rewriting Equation 2,

$$W'_{bal} = W' - \int_0^t W'_{\exp} \cdot e^{-\left(\frac{(t-u)}{\tau_{W'}}\right)} du$$

Treating W'_{exp} as a constant,

$$W'_{bal} = W' - W'_{\exp} \cdot \int_0^t e^{-\left(\frac{(t-u)}{\tau_{W'}}\right)} du$$

Integrating the exponential term with respect to u between limits $u = 0$ to $u = t$,

$$W'_{bal} = W' - \left[W'_{\exp} \cdot \left\{ \tau_{W'} \left(e^{\frac{u-t}{\tau_{W'}}} \right)_{u=0}^{u=t} \right\} \right]$$

$$= W' - W'_{\exp} \cdot \tau_{W'} \left(e^{\frac{t-t}{\tau_{W'}}} - e^{\frac{0-t}{\tau_{W'}}} \right)$$

Therefore,

$$W'_{bal} = W' - W'_{\exp} \cdot \tau_{W'} \left(1 - e^{\frac{-t}{\tau_{W'}}} \right)$$

Dimensional analysis of the integration of Equation 2

The result of the integration of Equation 2 is

$$W'_{bal} = W' - W'_{\exp} \cdot \tau_{W'} \left(1 - e^{\frac{-t}{\tau_{W'}}} \right) \qquad [6]$$

Units of the terms on the left-hand-side (LHS): W'_{bal} is in Joules (J)

Units of the terms on the right-hand-side (RHS): W' is in J, W'_{exp} is in J, $\tau_{W'}$ is in seconds (s). The exponential term is dimensionless as the unit of measurement for both t and $\tau_{W'}$ is s.

Therefore, the units will look like:

$$\mathrm{J} = \mathrm{J} - \mathrm{J}\cdot\mathrm{s}\cdot\left(1 - e^{\frac{\mathrm{s}}{\mathrm{s}}}\right)$$

$$\mathrm{J} = \mathrm{J} - \mathrm{J}\cdot\mathrm{s} \qquad [7]$$

The RHS cannot be computed due to the imbalance of the units.

Integration of Equation 3

Rewrtiting Equation 3,

$$W'_{bal} = W' - \int_0^t W'_{\exp}\cdot e^{-\left(\frac{(t-u)}{\tau_{W'}}\right)}dt$$

Treating W'_{exp} as a constant,

$$W'_{bal} = W' - W'_{\exp}\cdot\int_0^t e^{-\left(\frac{(t-u)}{\tau_{W'}}\right)}dt$$

Integrating the exponential term with respect to t between limits $t = 0$ and $t = t$,

$$W'_{bal} = W' - \left[W'_{\exp}\cdot\left\{\tau_{W'}\left(-e^{\frac{u-t}{\tau_{W'}}}\right)_{t=0}^{t=t}\right\}\right]$$

$$= W' - W'_{\exp}\cdot\tau_{W'}\cdot\left(-e^{\frac{u-t}{\tau_{W'}}} + e^{\frac{u-0}{\tau_{W'}}}\right)$$

$$= W' - W'_{\exp}\cdot\tau_{W'}\cdot\left(-e^{\frac{u-t}{\tau_{W'}}} + e^{\frac{u}{\tau_{W'}}}\right)$$

Therefore,

$$W'_{bal} = W' - W'_{\exp}\cdot\tau_{W'}\cdot e^{\frac{u}{\tau_{W'}}}\left(1 - e^{\frac{-t}{\tau_{W'}}}\right)$$

Dimensional analysis of the solution to Equation 3

The result of the integration of Equation 3 is

$$W'_{bal} = W' - W'_{\exp}\cdot\tau_{W'}\cdot e^{\frac{u}{\tau_{W'}}}\left(1 - e^{\frac{-t}{\tau_{W'}}}\right) \qquad [8]$$

Units of the terms on LHS: W'_{bal} is in J

Units of the terms RHS: W' is in J, W'_{exp} is in J, $\tau_{W'}$ is in s. The exponential term is dimensionless as the unit of measurement for u, t, and $\tau_{W'}$ is s.

Therefore, the units will look like:

$$J = J - J \cdot s \cdot e^{\frac{s}{s}} \cdot \left(1 - e^{\frac{s}{s}}\right)$$

$$J = J - J \cdot s \qquad [9]$$

The RHS cannot be computed due to the imbalance of the units, which is the same as the results of the dimensional analysis of Equation 2 in Equation 7.

Derivation of the recovery portion of Equation 5

The derivation below follows that presented by Skiba and colleagues (2) (Refer to Appendix 1 of (2))

Case: P < CP

Assumption: Rate of change of *W′* depends on the amount of *W′* remaining and the power output relative to CP. Additionally, the power below CP is assumed to be constant.

Therefore, the rate of change of *W′* is given by,

$$\frac{dW'}{dt} = \left\{1 - \frac{W'(t)}{W'_0}\right\} \cdot (CP - P)$$

Separating *dW′* and *dt* and substituting D_{CP} = CP – P,

$$\frac{dW'}{\left\{1 - \frac{W'(t)}{W'_0}\right\}} = D_{CP} \cdot dt$$

The limits of integration on both sides will be for a recovery interval starting at time "*u*" and ending at time "*t*". Therefore, the limits for *dW′* = *W′(u)* to *W′(t)*.

Integrating both sides,

$$\int_{W'(u)}^{W'(t)} \frac{dW'}{\left\{1 - \frac{W'(t)}{W'_0}\right\}} = \int_u^t D_{CP} \cdot dt$$

Integrating by treating ($1/W'_0$) as a constant,

$$\left[-W'_0 \cdot \ln\left\{W'_0 - W'(t)\right\}\right]_{W'(t)=W'(u)}^{W'(t)=W'(t)} = D_{CP} \cdot (t)_{t=u}^{t=t}$$

Multiplying both sides by ($1/W'_0$) and applying the limits on the RHS,

$$\ln\left[W'_0 - W'(t)\right]_{W'(t)=W'(u)}^{W'(t)=W'(t)} = \frac{-D_{CP} \cdot (t-u)}{W'_0}$$

Applying the limits on the LHS,

$$\ln\left\{W'_0 - W'(t)\right\} - \ln\left\{W'_0 - W'(u)\right\} = \frac{-D_{CP}\cdot(t-u)}{W'_0}$$

Using the laws of logarithms,

$$\ln\left\{\frac{W'_0 - W'(t)}{W'_0 - W'(u)}\right\} = \frac{-D_{CP}\cdot(t-u)}{W'_0}$$

Taking exponential on both sides,

$$\frac{W'_0 - W'(t)}{W'_0 - W'(u)} = e^{\frac{-D_{CP}\cdot(t-u)}{W'_0}}$$

Multiplying by $W'_0 - W'(u)$ on both sides and rearranging,

$$W'(t) = W'_0 - \left\{W'_0 - W'(u)\right\}\cdot e^{\frac{-D_{CP}\cdot(t-u)}{W'_0}}$$

Replacing $W'(t)$ with W'_{bal}, $\{W'_0 - W'(u)\}$ with W'_{exp}, and (t-u) with t gives,

$$W'_{bal} = W'_0 - W'_{\exp}\cdot e^{\frac{-D_{CP}\cdot t}{W'_0}} \qquad [10]$$

Equation 10 is identical to the expression representing the recovery portion in Equation 5.

References

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